



## The History of Pascal's Triangle

In a lot of the Western world, the triangle is called Pascal's triangle. It was named after a 17th century mathematician, Blaise Pascal, who developed many of its properties and applications in his book *Treatise on the Arithmetical Triangle* (*Traité du triangle arithmétique*).

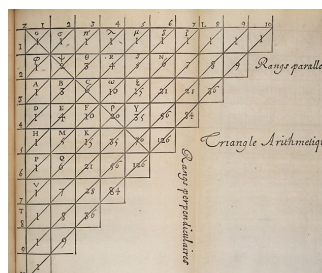


Figure 1: Retrieved from [Know-It-All to Know-Nothing](#).

However, the triangle was discovered and studied hundreds of years before Pascal was even born. As a result, the triangle has been known by many different names around the world.

Pascal's triangle is also known as the **Staircase of Mount Meru** in India, **Khayyam Triangle** in Iran, **Yang Hui's Triangle** in China, and **Tartaglia's Triangle** in Italy.

It has also been known as the **figurate triangle**, the **combinatorial triangle**, and the **binomial triangle**.

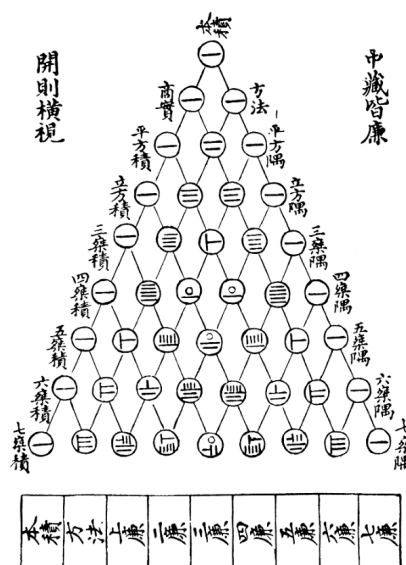


Figure 2: Retrieved from [Wikipedia](#).

## Properties of Pascal's Triangle

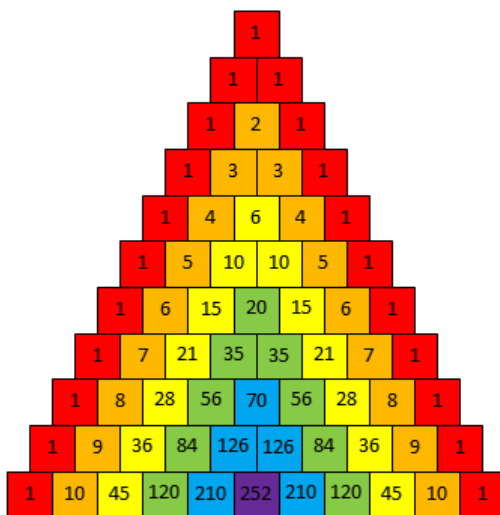
### Stop and Think

Do you see any patterns in Pascal's Triangle?



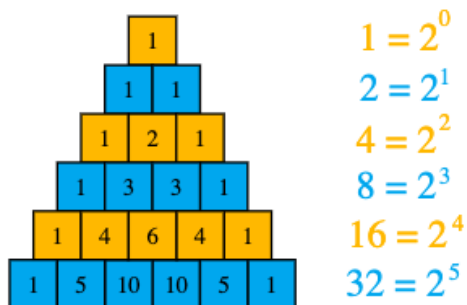
## Property 1: Symmetry

Pascal's triangle is **symmetrical**, that is, the entries on the left side are the same as the entries on the right side.



## Property 2: Row Sums

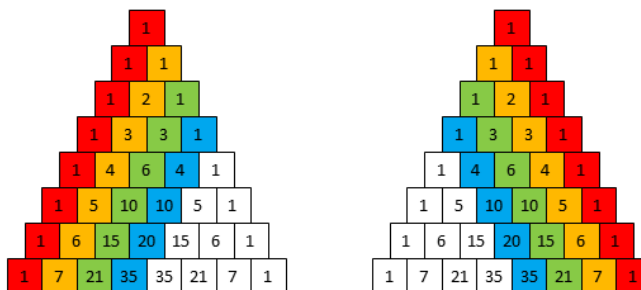
Looking at the sum of the entries in each row of Pascal's triangle, we can see that the sum doubles moving down the rows. That is, the **row sums** are all **powers of 2**. More precisely, each row sum is  $2^n$  where  $n$  is the row number. Remember that  $2^n$  is the product of  $n$  2s, that is,  $2^n = \underbrace{2 \times 2 \times \dots \times 2}_{n \text{ times}}$ .



### Exercise 2

What is the sum of the entries in row 7 of Pascal's triangle?

### Property 3: Diagonals



#### First Diagonal

The first diagonal of Pascal's triangle contains only the value **1**. These entries are along the edge of the triangle and each only has one entry directly above it (containing a 1) all the way back to the top 1.

#### Second Diagonal

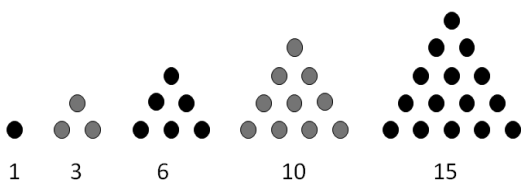
The second diagonal of Pascal's triangle contains the **counting numbers**, also known as **natural numbers**. Each of these entries is also equal to its row number.

#### Third Diagonal

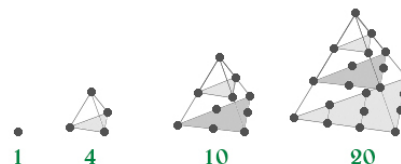
The third diagonal of Pascal's triangle contains the **triangular numbers**. These are the numbers which form increasing sizes of triangles.

#### Fourth Diagonal

The fourth diagonal of Pascal's triangle contains the **tetrahedral numbers**. These are the numbers which form increasing sizes of tetrahedrons, which are triangular pyramids with congruent equilateral triangles for each face.



Triangular numbers



Tetrahedral numbers

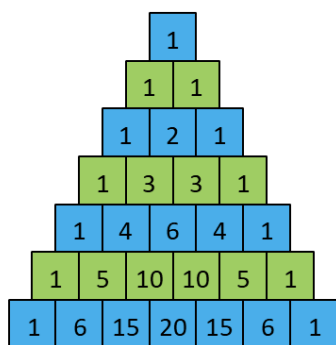
Figure 3: Retrieved from [GeeksforGeeks](#).



### Property 4: Powers of 11

For each row, the sum of each entry multiplied by a power of 10 whose exponent is the term number is equal to a **power of 11** where the exponent is the row number.

For example, in the second row, the first 1 has term number 0, the 2 has term number 1, and the last 1 has term number 2, so  $1(10^0) + 2(10^1) + 1(10^2) = 1 + 20 + 100 = 121 = 11^2$ . Recall that  $10^0 = 1$ .



$$\begin{aligned}
 &1(10^0) = 1 = 11^0 \\
 &1(10^0) + 1(10^1) = 11 = 11^1 \\
 &1(10^0) + 2(10^1) + 1(10^2) = 121 = 11^2 \\
 &1(10^0) + 3(10^1) + 3(10^2) + 1(10^3) = 1331 = 11^3 \\
 &1(10^0) + 4(10^1) + 6(10^2) + 4(10^3) + 1(10^4) = 14641 = 11^4 \\
 &1(10^0) + 5(10^1) + 10(10^2) + 10(10^3) + 5(10^4) + 1(10^5) = 161051 = 11^5 \\
 &1(10^0) + 6(10^1) + 15(10^2) + 20(10^3) + 15(10^4) + 6(10^5) + 1(10^6) = 1771561 = 11^6
 \end{aligned}$$

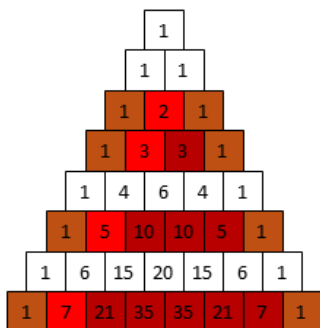
#### Exercise 3

Use Pascal's triangle to find  $11^7$ .

### Property 5: Prime Rows

A **prime number** is a positive whole number greater than one that has only two positive factors: one and itself.

If term 1 in a row is a prime number, then all other entries in that row, other than the two 1s, are multiples of that prime.

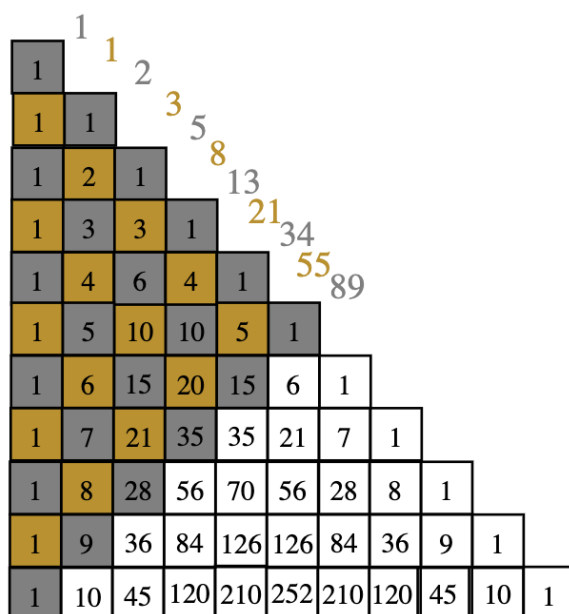




## Property 6: Fibonacci Sequence

The **Fibonacci Sequence** is a pattern of numbers where the first two numbers are 1 and 1 and every other number is the sum of the previous two. So, the first ten Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55.

If Pascal's triangle is shifted to the left, then the sums of the entries in the new diagonals will be the numbers in the Fibonacci Sequence.



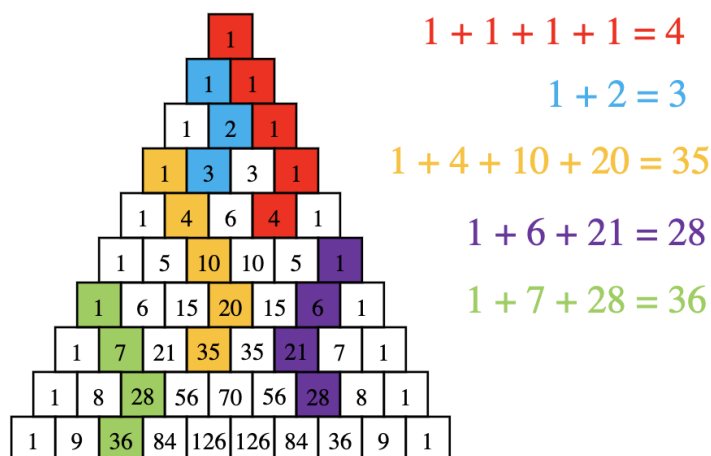
### Exercise 4

What is the 12<sup>th</sup> Fibonacci number?



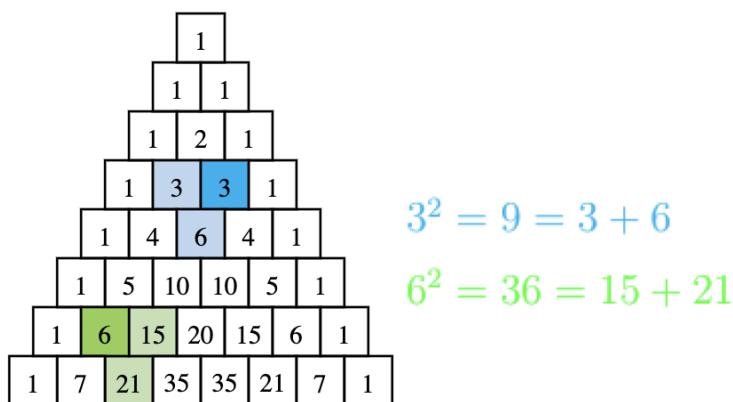
### Property 7: Hockey Stick Pattern

Starting at any entry which is a 1 and moving down the triangle diagonally, then changing the direction of the diagonal for the last entry results in something that looks like a **hockey stick**. The sum of the numbers on the diagonal is equal to the entry which is not on the diagonal.



### Property 8: Squares

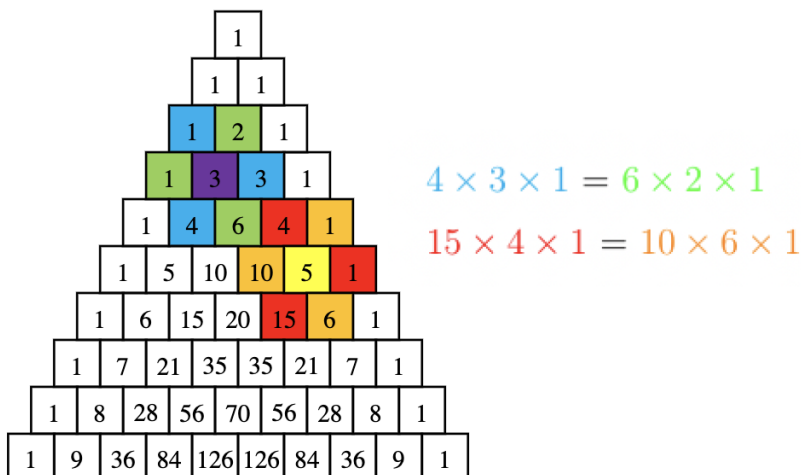
For the second diagonal, the one containing the counting numbers, the square of any entry is equal to the sum of the entry beside it which is in the third diagonal and the entry below both of those. Notice that there is no entry which is in the third diagonal that is beside the 1 in the second diagonal, so in this case we count that value as 0 and so we'd have  $1^2 = 1$ .





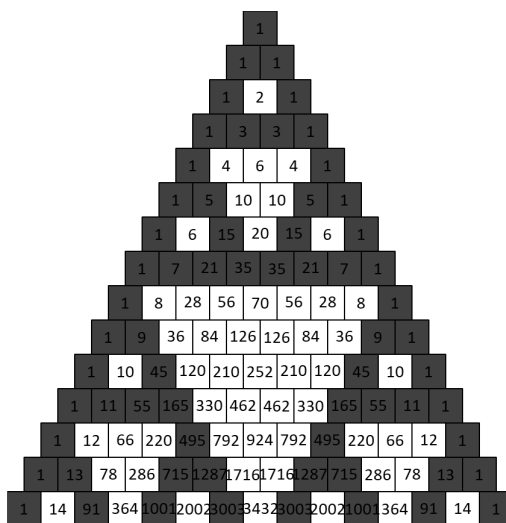
### Property 9: Hexagons

Each entry which isn't a 1 has six neighbours, that is, six other entries which are directly touching it. These numbers can be split into two groups of three numbers where the numbers within a group are not neighbours. The products of multiplying the numbers in both groups are equal.



### Property 10: Sierpinski Triangle

By colouring out all the odd entries in Pascal's triangle, we uncover a special pattern which is called Sierpinski's Triangle.







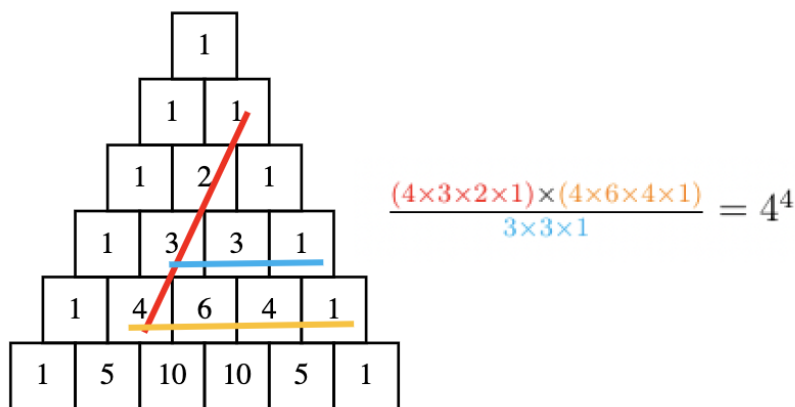
This pattern is called a fractal, it is created by a repeating pattern, in this case equilateral triangles.



Figure 1: Retrieved from [vecteezy](#).

**Property 11:**  $n^n$

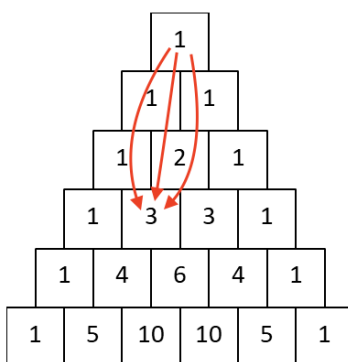
Starting from an entry in the second diagonal containing the number  $n$ , the product of the individual products of both the entries from  $n$  to 1 along the second diagonal and the entries from  $n$  to 1 along the horizontal row (every entry in the row excluding a 1) divided by the product of the entries in the row above (excluding a 1), will be equal to  $n^n$ .



**Exercise 5**  
Use Pascal's triangle to find  $6^6$ .

## Property 12: Paths

A **path** in Pascal's triangle starts at the top 1 and move downwards through cells that are directly touching, that is, adjacent. Each entry is the number of different paths from the top 1 to that entry. For example, in the image below, there are three different paths from the top 1 to the cell with row number 3 and term number 1.



### Exercise 6

Below is a checkerboard with one checker within a square on the bottom row. The checker can only move diagonally to a square above it. How many different ways can the checker move to the opposite side of the board?

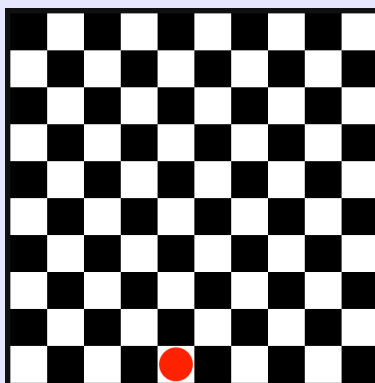


Figure: Retrieved from [Wikipedia](#).



## Combinations and Pascal's Triangle

Before we can see how combinations relate to Pascal's triangle, we must first review some concepts from the Week 2 Counting lesson.

Define  $0! = 1$  and  $n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$  for  $n > 0$  where  $n$  is a whole number. The expression  $n!$  is read as " **$n$  factorial**".

Note that each factorial can be found using the previous factorial:  $n! = n \times (n - 1)!$ .

**Combinations** is a method of counting how many ways there are to choose  $k$  objects from  $n$  objects where there is no repetition and order does not matter. We use the notation  $\binom{n}{k}$  which is read as " $n$  choose  $k$ ". To find the number of combinations we have the following formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Combinations are largely connected to Pascal's triangle. In fact, any entry of Pascal's triangle can be found using combinations.

Any entry of Pascal's triangle is equal to  $\binom{n}{k}$  where  $n$  is the row number of the entry and  $k$  is the term number of the entry.

For example, by looking at Pascal's triangle, we know that the entry which is term 2 of row 3 is equal to 3, and using the choose formula we get  $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2} = 3$ .

### Exercise 7

Find the entry of Pascal's triangle given each row number and term number.

a) row 0, term 0

c) row 4, term 4

e) row 28, term 0

b) row 6, term 3

d) row 9, term 1

f) row 8, term 7

Since every entry can be found using combinations, we can rewrite Pascal's triangle by replacing the entries with the choose notation.

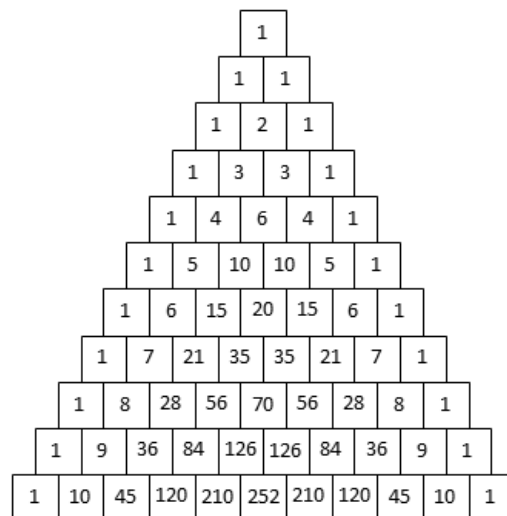
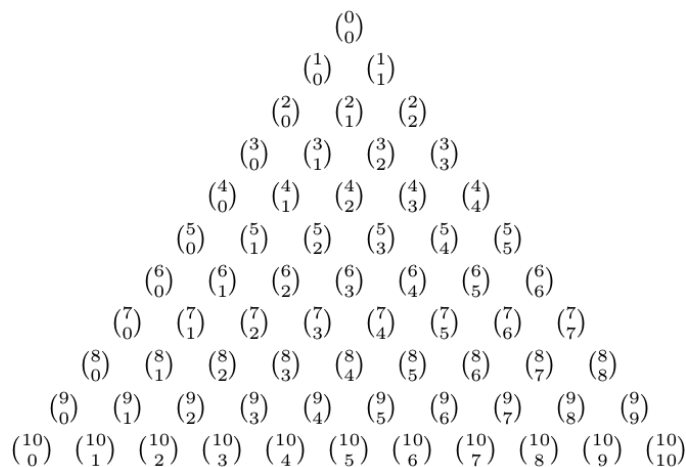


Figure 2: Retrieved from [StackExchange](#).

We can now use Pascal's triangle, instead of the choose formula, to solve some combinations problems.

**Example 1**

There are 9 students in a Math Club. How many ways are there to choose 5 students to help with setting up for their Pi Day event?

We want to choose 5 people out of 9 people. The number of ways of doing this is equal to  $\binom{9}{5}$ . Looking at Pascal's triangle, we see that the entry with row number 9 and term number 5 is 126. So, there are 126 different ways of choosing 5 students to help with set up.

**Exercise 8**

Audrey is painting her room and she has 6 different paint colours: red, orange, yellow, green, blue, and purple. Use Pascal's triangle to determine how many different colour combinations can she use to paint her room if she uses...

- a) 1 colour
- b) 4 colours
- c) 6 colours